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# REVIEWS

**Lectures on Fluid Mechanics.** By MARVIN SHINBROT. Gordon & Breach, 1973. 222 pp. $14.25.

This book is, I believe, the first introductory text in English to lay before the reader samples both of the traditional, concrete approach to fluid dynamics (familiar to readers of this *Journal*) and of the modern, more abstract theory (whereby the exact solution of a rather general problem in viscous flow, say, is first secured as a point in an appropriate function space, the members of which may have distinctly fewer derivatives than appear in the Navier–Stokes equations). The author's purpose is "to introduce the mathematically sophisticated listener to some of the problems of the mechanics of an incompressible fluid", and he assumes that "the reader is familiar with functional analysis, some complex variables, and little else". It turns out, however, that the reader need have only the most primitive ideas of functional analysis; what *is* essential for the final chapters is a taste for delicate, and occasionally elaborate, inequalities between integrals.

The book is in two parts. Part I, *Setting the scene,* is 132 pages long, and begins by dipping into statistical mechanics to derive the equations of motion, with only a sketch of the continuum viewpoint. This is followed by five chapters on potential flow, with perhaps more emphasis on general theorems than is found in conventional texts of comparable length, but containing also the usual examples of Rankine's combined vortex (with a free surface) and of the ideal flows past sphere, circle and Joukowski aerofoil. Next, there are three chapters on elementary aspects of viscous flow: the first derives the stress/rate-of-strain relation and the energy equation; the second gives examples of parallel flows and of the Jeffery–Hamel solutions for source flow between inclined planes; and the third is a very brief description of the Stokes, Oseen and boundary-layer approximations.

Part II, *A taste of the modern theory,* is 83 pages long and deals with the initial-value problem of the Navier–Stokes equations, in which one asks what happens to fluid of uniform density and viscosity within a fixed container, when the initial velocity field (at time $t = 0$) is known. In the present treatment, the container is a bounded domain $V$ in the (real, Euclidean) three-dimensional space $R^3$, with a smooth boundary $\partial V$ on which the velocity is zero for all $t \geqslant 0$. The preliminary chapter 10 introduces some of the mathematical apparatus; the next three proceed more or less as follows. The author begins with his own ingenious derivation, by means of finite differences with respect to the time variable, of the celebrated weak solution which E. Hopf constructed in 1951 by a dazzling application of Galerkin's method.† (Hopf followed the path opened by Leray in

† In other words, by means of approximations

$$u^k(x, t) = \sum_{\nu=1}^{k} a^\nu(x)\beta_{k,\nu}(t)$$

to the velocity; here the $a^\nu$ are 'basis' vector fields, solenoidal and vanishing on the boundary $\partial V$, and in principle known *a priori*, while the scalars $\beta_{k,\nu}$ are found as the solutions of $k$ simultaneous *ordinary* differential equations.

1933–4.) For three space dimensions, this weak solution is the only one known to exist for all time; but it is not required to have a time derivative even in the generalized sense, and may not be unique; as far as is known, the corresponding total energy at any instant may be *less* than the initial energy minus that spent by dissipation. Next, we learn that, if in some time interval $[0, T)$ this weak solution should happen to be just a little smoother than has been proved, then it would be unique for such times and would satisfy the energy equation. With still better differentiability properties (in fact, when the velocity has in a generalized sense the derivatives appearing in the Navier–Stokes equations), a weak solution earns the name *strong*; and we are now led to a result first proved by Kiselev and Ladyzhenskaya, and later sharpened by Kaniel and Shinbrot. This shows that, when the initial velocity field is not too rough, a strong solution, which is therefore unique, exists for sufficiently small times; alternatively, if the initial velocity is sufficiently small, such a solution exists for all time. (Here 'sufficiently small' means 'pathetically small' from the practical point of view.) The plot thickens further with the description (due to Leray for fluid occupying the whole space $R^3$, and to Kaniel and Shinbrot for bounded domains $V$) of the set of instants $t$ at which a weak solution can seriously misbehave itself: this set is bounded, is the complement of a union of half-open intervals, and has Lebesgue measure zero!

The final chapter 14 deals with a reproductive property of the Navier–Stokes equations: for any sufficiently small body force, one can find (in a bounded domain $V$) an initial velocity field that is repeated at a prescribed time $t_1 > 0$.

Prof. Shinbrot has made significant contributions not only to the foregoing story, but also to the theory of water waves, to the study of the no-slip boundary condition from the viewpoint of kinetic theory with diverse reflexion laws, and to various matters of Analysis. Accordingly, he is exceptionally well qualified to write a book on both sides of the theory of fluid motion; but his text is marred by what I can only imagine to be extreme haste and carelessness in its preparation. There is a host of small inaccuracies, ranging from (*a*) the omission of the large factor $N$, representing the number of mass-points, from the definitions of density and mean velocity in chapter 1, to (*b*) at least six wrong exponents in the mathematical tool-kit of chapter 10. In addition, the following howlers may dismay not only the mathematically sophisticated reader to whom the book is specifically addressed.

(i) In the proof of the transport theorem on p. 9, and repeatedly thereafter, it is claimed that certain maps (essentially those from material to spatial co-ordinates) are one-to-one merely because their Jacobians are strictly positive everywhere. The domains of these maps are specified only by the ambiguous phrase "transformations in $R^n$". When that domain is a proper subset of $R^n$ (a case that we certainly need in applications), positivity of the Jacobian is *not* enough.

(ii) Lemma 2.3.3, p. 39, which asserts sufficient conditions for the existence of a (single-valued) velocity potential, is conspicuously false in both statement and proof for multiply connected domains, despite remarks immediately before the lemma that suggest an awareness of the danger.

(iii) On p. 42, the potentials of a point dipole and of the classical flow past a sphere are repeatedly and consistently wrong.

(iv) The Hilbert space $L^2(V \to R^3)$, of vector-valued functions defined and square-integrable on a (possibly unbounded) domain $V$ in $R^3$, has an orthogonal decomposition into (*a*) the closure in $L^2$ of smooth solenoidal vector fields that vanish on the boundary $\partial V$, and (*b*) gradients $\nabla f$ of single-valued scalar functions (even when $V$ is multiply connected). This decomposition is central to Navier–Stokes theory, and is a particular favourite of Prof. Shinbrot, but the proof of it here is irreparably wrong at almost the first step: in general, the extension of $\phi$ that precedes equation (2.1) on p. 141 does not yield a function in what is called $(\mathscr{L}^2(R^3))^{\perp}$ here. (The final step of the proof is also false, but that could be repaired if the earlier part were correct.)

So much for actual mistakes; but my complaints do not end there. An excursion into statistical mechanics is certainly to the good, but, as the author admits, that subject cannot at present provide a firm and unambiguous foundation for the continuum theory, let alone by simple arguments. Such a foundation, necessarily depending on strong hypotheses about the map from material points (in an arbitrary set) to spatial points, never appears in the book, and a certain loss of clarity results. Indeed, this fundamental map is never given a name: whether we are in the phase space $R^{6N}$ or in $R^3$, the same letter is used for the generic symbol of points and for the function of time and initial position that describes the path of a marked point.

In Part II, which seems closer than Part I to the author's heart and mind, lemmas and proofs tend to have less direction and edge than one might wish, in two respects. First, the standard properties of mollified functions, $L^p$ spaces and Sobolev spaces, which are now available in a number of texts, appear in an unnatural order and do not always precede the more special, and sometimes elaborate, estimates needed for Navier–Stokes problems. Second, key steps are not given due prominence relative to routine ones. For example, the cornerstone of chapter 12 is a Machiavellian bound for integrals of the form

$$\int_0^{t_1} dt \int_V u \cdot (v \cdot \nabla)\, w\, dx,$$

when the vector fields $u(x,t)$, etc., are restricted in a certain way. With this estimate emphasized as a preliminary lemma (as it is not in the book), and with an economical notation for mollified functions, the proofs in chapter 12 can be made significantly shorter and more transparent.

A minor irritant is that the symbol $q$ denotes the position vector in $R^3$, and $s$ the velocity vector, throughout the book. After a few days' practice, one learns to accept this; but the mere reading of equations becomes a severe strain when, within the same equation, different velocity fields are denoted by $s_1$ and $s_2$ (instead of the usual $u$ and $v$ or $\mathbf{u}$ and $\mathbf{v}$) and, at the same time, inner-product brackets ( , ) embrace a thicket of argument brackets ( ), mollified functions being written at full length.

Despite these criticisms, I am happy to have read, and to own, the book. For a relatively short work in which no prior knowledge is assumed, it contains much that is not readily, or so palatably, available elsewhere, and I have learned a great deal. The author makes good his promise in the Introduction to point constantly to unsolved problems, and his characteristic remark that "this work can be used as the beginnings of a sourcebook for problems for Ph.D. theses" is scarcely exaggerated. The book is written with a lively style, a highly individual point of view, total irreverence towards accepted conventions and beliefs, and a critical sense that is often penetrating. Above all, it is never dull; Prof. Shinbrot has a remarkable ability to sharpen our curiosity, and to freshen our appetite for the mathematical problems of fluid mechanics.

L. E. FRAENKEL

**Analytical Methods in Planetary Boundary-Layer Modelling.** By R. A. BROWN. Hilger, 1974. 148 pp. £8.00.

If the reader expects from the title of this book to be able to gain from it a good understanding of the planetary boundary layer, or of how to model it, he is going to be disappointed. The atmospheric boundary layer is a region in which moist (and dry) convection is important, and which undergoes large diurnal variations. The oceanic boundary layer is a region which is better mixed than the region below, which is strongly affected by storms, and which changes a great deal seasonally. Yet buoyancy and time-dependent effects are given very little space in the book. The author states his policy on this matter in the preface:

> 'It is recognized that almost all planetary boundary layers are diabatic to some degree. Thus, practical application demands consideration of the thermal effects. In fact, it is apparent that the average planetary boundary layer is also time dependent. There are no analytic solutions incorporating all aspects of the observed flows. The boundary layer problem has been attacked from three fairly independent viewpoints: the dynamic flow problem for the neutral layer; the thermal convection problem; and the numerical integration of the complete equations. There is a large quantity of literature on the first two topics, and a rapidly growing amount in the third category. Mastery of the planetary boundary layer problem will require study of all three segments.
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> This text presents the first approach. A presentation devoted to analytical treatments must emphasize the dynamic flow (as contrasted to the convective flow) as it emerges from the neutrally stratified solutions. It is felt that the methods presented herein have didactic value which is indispensable to an understanding of the planetary boundary layer. Analytic solutions, and hence the neutrally stratified case, provide invaluable touchstones for an eventual development of the diabatic problem.
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> This text is written from the geophysical – primarily meteorological – point of view.'

What the book does do is to introduce ideas used for modelling the mean velocity profile in the atmospheric boundary layer (56 pages, chapters 4–8) and also the large eddies or secondary flow (26 pages, chapters 9 and 10). The arrangement of the material is clear and the account readable. Detailed mathematics is avoided by simply describing the 'analytical methods' and presenting relevant equations. A summary of the material covered is given below.

After three introductory chapters, the classical Ekman solution is presented (chapter 4) as a constant eddy viscosity solution of the equations. Chapter 5 is about the surface layer. The ideas behind the logarithmic velocity profile are introduced and the Monin–Oboukhov similarity theory about thermal effects is presented, together with empirical fits to the similarity profiles. The next three chapters, on the other hand, are about modelling the complete boundary layer in neutral conditions. Chapter 7 is about the appropriate similarity solutions while chapter 8 is about solutions for prescribed piecewise continuous variations of eddy viscoscity with height. Chapter 6 is really about the limiting case of one of these solutions but this is obscured by a misguided appeal to asymptotic matching theory. Chapters 9 and 10 may be considered an attempt to model the large eddies in the planetary boundary layer and to explain associated cloud patterns (some satellite pictures are included). The basis of this modelling is the stability theory for a classical Ekman layer in which there may be a uniform temperature gradient. This gives the eddy profiles and another condition is used to determine amplitudes. The subject is one to which the author of the book has made a considerable contribution himself. Finally (chapters 11 and 12), a brief allusion is made to thermal and time-dependent effects.

Even within the limited scope of the book, there is a lot of relevant work which has not been mentioned, such as the attempts to close the equations by methods other than prescribing an eddy viscosity distribution or by explicit inclusion of large eddies. Perhaps the book will be of most use to people who want a short, well laid-out, readable introduction to concepts and methods used in modelling the velocity profile in the atmosphere.

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